Fundamental of

- Textbook: Applied Electromagnetics, Fawwaz T. Ulaby, 5th edition

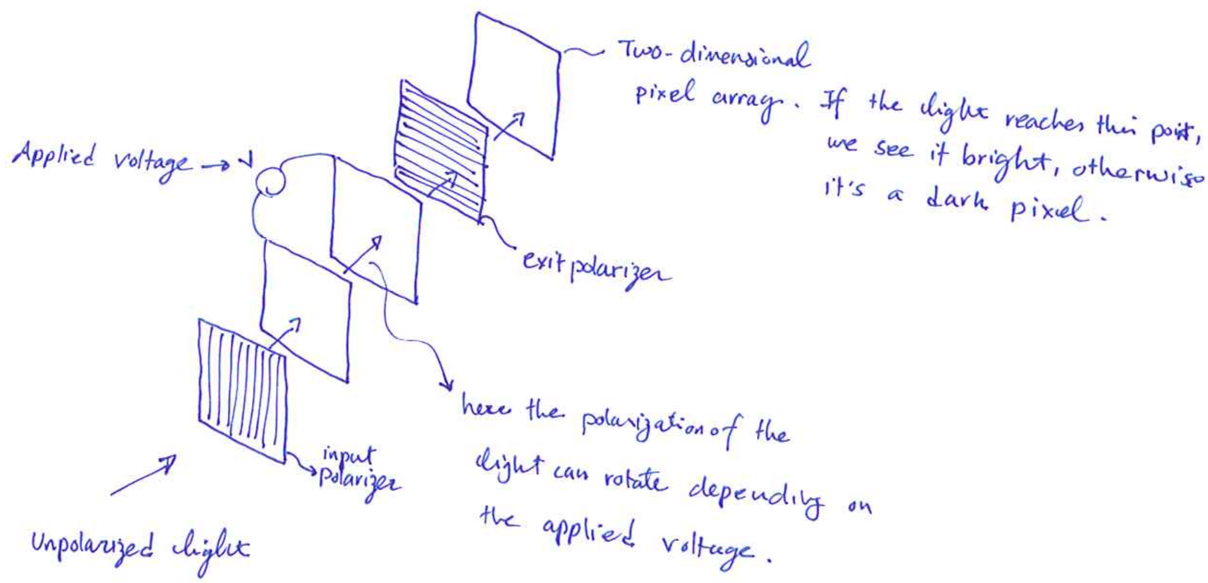
- Grading procedure: Exam#1: 20%, Exam #2: 20%, Final 30%, HW 30%

- Office hour

## Introduction:

Electromagnetics is at the heart of electrical & computer engineering.

Example: Liquid Crystal Display (LCD)




## The Nature of Electromagnetism:

There are Four fundamental forces in nature:

1) Nuclear force: strongest, short range in nuclei

2) Weak interaction force: $10^{-14}$ of nuclear force, interaction between elementary particles.

3) Electromagnetic force: $10^{-2}$ of nuclear force, between charged particles, atoms & molecules

4) Gravitational force: weakest, $10^{-41}$ of nuclear force, dominant in macroscopic systems such as solar system.

Electromagnetic force Consist of Electrical force $F_e$, and magnetic force $F_m$.

## Electric Field:

The souce is electric charge. Fundumental quantity of charge is that of a single electron:

$$\boxed{e = 1.6 \times 10^{-19} \quad (C)}$$

C is "Coulomb" in honor of the 18th century french scientist Charles Augustin de Coulomb.

Electron charge: $q_e = -e$

Proton charge: $q_p = e$

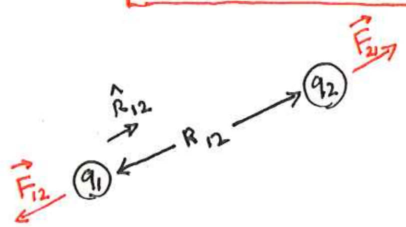
Two properties:
1) charges can be added. e.g. $q = 3q_e + 5q_p = 2e$
2) charge is conserved (is not annihilated or created)

Coulomb's law: Force between two charges of $q_1$ & $q_2$ is given by:

$$\boxed{\vec{F}_{21} = \frac{q_1 q_2}{4\pi\varepsilon_0 R_{12}^2}\hat{R}_{12} \; (N)} \quad \text{(in free space)}$$

$R_{12}$: distance between $q_1$ & $q_2$

$\varepsilon_0$: electrical permitivity of free space $8.854 \times 10^{-12}$ (F/m)

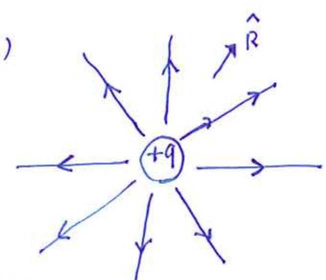



$\vec{F}_{12} = -\vec{F}_{21}$

## Electric Field:

Electric field intensity $\vec{E}$ due to a charge $q$ is given by:

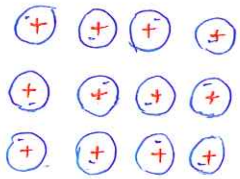
$$\boxed{\vec{E} = \frac{q}{4\pi\varepsilon_0 R^2}\hat{R} \quad \left(\frac{V}{m}\right)} \quad \text{(in free space)}$$




□ What happens to the electric field inside a material?

Assume we place a positive point charge in a material composed of atoms.

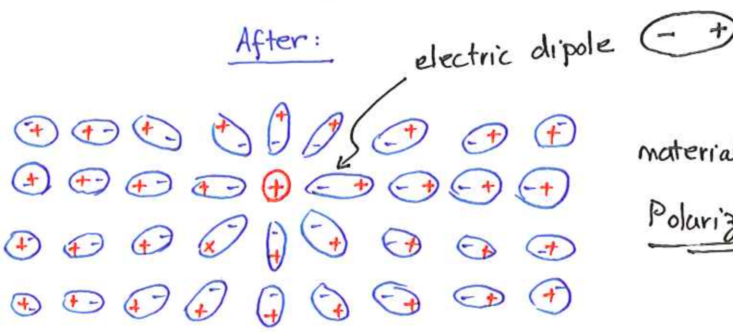
Before:



The material is electrically neutral.

Add a ⊕ charge inside the material

After:




material is Polarized.

The degree of polarization depends on the distance from the isolated point charge.

Polarization is the process of electric dipole formation.

Dipoles of the atoms (or molecules) counteract the field due to the point charge $\Rightarrow$ therefore the electric field inside the material is different from that in free space.

To extend our equation for electric field from free space to inside a material we replace $\varepsilon_0$ by $\varepsilon$ : $\boxed{\varepsilon = \varepsilon_r \varepsilon_0}$ $\varepsilon_r$ depends on the material propeties.

$$\boxed{\vec{E} = \frac{q}{4\pi \varepsilon R^2} \hat{R}}$$

$\varepsilon_r$ is unitless $\Rightarrow$ $\varepsilon$ has same unit as $\varepsilon_0$ (F/m)

$\varepsilon_r$ is called "relative permittivity" or "dielectric constant" of the material.

- for vacuum : $\varepsilon_r = 1$ .
- For air near Earth's surface $\varepsilon_r = 1.0006$
- For Teflon $\varepsilon_r = 2.1$
- For wood 1.5 – 4
- For Distilled water 81

## Electric flux density, D:

It is convenient to use a new quantity that doesn't depent the material's property by multiplying E by $\varepsilon$:

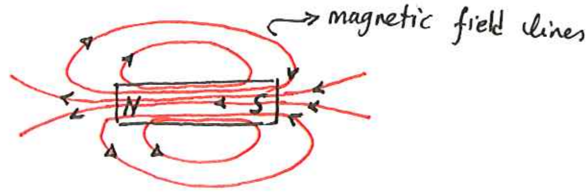
$$\varepsilon \times E = \frac{q}{4\pi \varepsilon R^2} \times \varepsilon \rightarrow \boxed{\vec{D} = \varepsilon \vec{E} = \frac{q}{4\pi R^2} \hat{R} \quad \left(\frac{C}{m^2}\right)}$$

→ D depends only on charge q and distance R.

## Magnetic Fields

First magnetic stones were discovered by Greeks 800 B.C.. We call these stones now "magnetite" ($Fe_3O_4$).

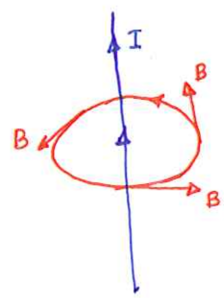
Each magnet has two poles : North & South poles.



It is impossible to separate N and S. If the magnet is broken into pieces, each piece has its own N & S poles.

## Magnetic flux density, B

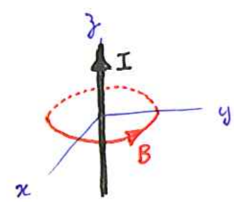
Magnetic flux density, B, exists around magnets. It can also be created by electric current. This connection was first discovered by Danish scientist Hans Oersted in 1819.




A wire carrying current I, creates a magnetic field $\vec{B}$.

Shortly after Oersted, French scientists Jean Baptiste Biot and Felix Savart developed an expression relating I and B. It is now called "Biot_Savart law":

$$\boxed{\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi} \quad (T)}$$




The unit is Tesla. $\mu_0$ is magnetic permeability for free space:

$$\boxed{\mu_0 = 4\pi \times 10^{-7} \quad (H/m)}$$

There exists a relation between the speed of light c, and $\varepsilon_0$ & $\mu_0$:

$$C = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3\times 10^{8} \quad (m/s)$$

Permeability $\mu$ accounts for "magnetization" properties of a material. A nonmagnet material has $\mu = \mu_0$ and a ferromagnetic material like iron has $\mu \gg \mu_0$.

$\mu$ is defined as: $\boxed{\mu = \mu_r \mu_0 \quad (H/m)}$ (in analogy to relative permitivity)

$\mu_r$ is dimensionless called relative permeability.

## Magnetic Field Density

We also define a magnetic field density that doesn't depend on $\mu$ by deviding B by $\mu$:

$$\frac{1}{\mu} \times B = \hat{\phi} \frac{\mu I}{2\pi r} \times \frac{1}{\mu} \Rightarrow H = \frac{B}{\mu} = \hat{\phi} \frac{I}{2\pi r} \quad \text{or} \quad \boxed{B = \mu H}$$

## Static and Dynamic Fields

Charge, $q \Longrightarrow$ Electric field. So if the charge doesn't chang with time, E remains constant. This corresponds to "Electrostatics": $\frac{\partial q}{\partial t} = 0 \Rightarrow$ E is constant.

Current, $I \Rightarrow$ Magnetic field. So if I is constant, B is constant. This corresponds to "Magnetostatics": $\frac{\partial I}{\partial t} = 0 \Rightarrow$ B is constant.

Now consider the case where I changes with time. Since $I = \frac{dq}{dt}$, it means that the amount of charge present in a given section of the wire varies also with time. So E varies with time.

In general: A time-varying electric field will generate a time-varying magnetic field, and vice versa.

## Conductivity, $\sigma$ ($S/m$):

The conductivity indicates how easy charges move in a material. $\sigma = 0$ means material is insulator and charges cannot move inside this material. This is also called "perfect dielectric".

If $\sigma = \infty$, the material is superconductor and charges (electrons) can move perfectly free inside the material. This is also called "perfect conductor".

$\varepsilon$, $\mu$, and $\sigma$ are the "constutive parameters" of a material.

A "homogenous" material has constant constutive parameters throughout the medium.

# Traveling Waves

Examples: Ocean waves, Sound waves, electromagnetic waves that constitute light earthquake waves, etc.

- waves carry energy
- waves have velocity → light is EM wave with speed $3\times10^8$ m/s
- Some wave are "linear": they pass right through each other and when is overlap, they add up.

  Sound waves & EM waves are linear. Water waves are approximately linear.
- waves can be:
  - "Transient", i.e. short-duration disturbance.
  - "Continous", i.e. generated by an oscillating source.

Examples:

1D: One-dimensional wave:

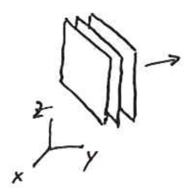

Traveling on a string.

2D: Two-dimensional wave:

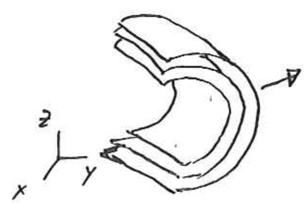

circular wave of ripples on a pond.

3D: Three-dimensional wave:

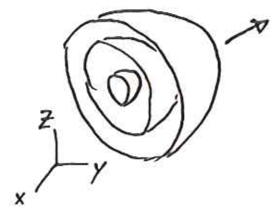
Plane_wavefront




Circular_wavefront




spherical_wavefront




# Sinusoidal Wave in a lossless Medium

- Lossless medium: Wave amplitude does not attenuate within it or on its surface.

General form for a 1D Sinusoidal wave:

$$\boxed{y(x,t) = A\cos\left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda} + \phi_0\right)} \quad (m)$$

$A$: amplitude

$T$: time period

$\lambda$: special wavelength

$\phi_0$: reference phase

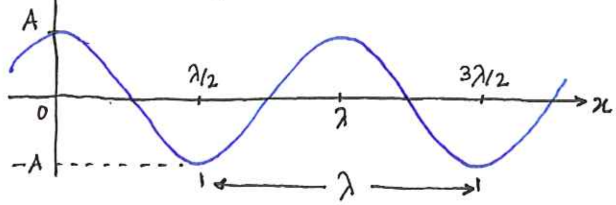
– we may also write:

$$y(x,t) = A\cos\varphi \qquad \text{where } \varphi = \frac{2\pi t}{T} - \frac{2\pi x}{\lambda} + \varphi_0 \quad \text{(rad)}$$

Phase

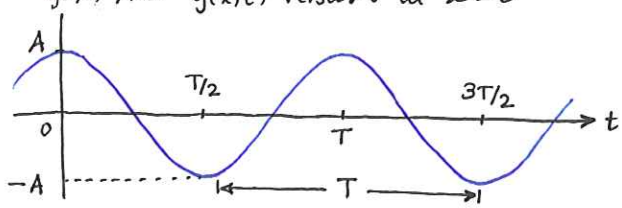
– Consider $y(x,t) = A\cos\left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda}\right)$, the case when $\varphi_0 = 0$:

$y(x,0)$, i.e. $y(x,t)$ versus $x$ at $t=0$



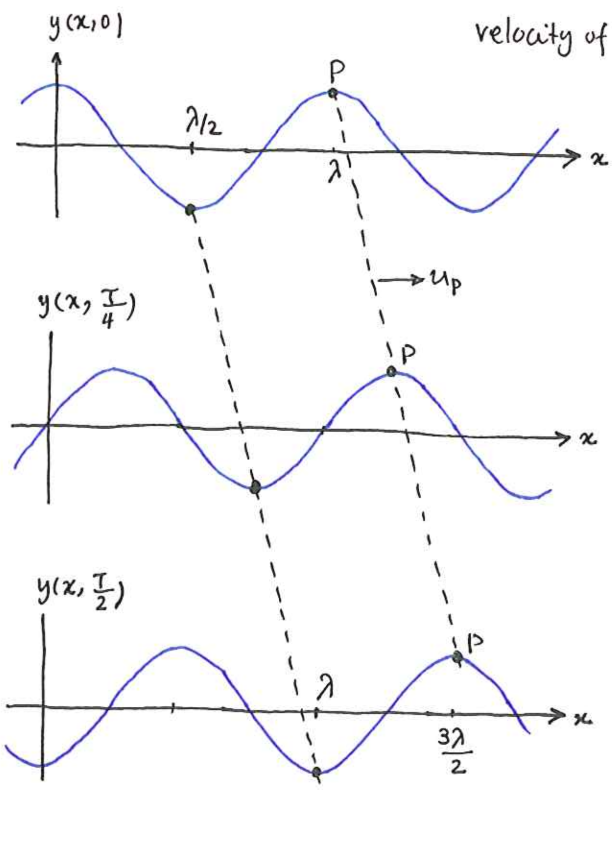

← snapshot at $t=0$

$y(0,t)$, i.e. $y(x,t)$ versus $t$ at $x=0$




If we take snapshots at different times, say $t = 0, \frac{T}{4}, \frac{T}{2}$, we see that the wave is moving forward with a velocity $\boxed{u_p = \frac{\lambda}{T}}$ :

$u_p$ is called "phase velocity" or sometimes as "propagation velocity", i.e. the velocity of the wave pattern, and NOT the medium.




Say for point P:

$$\Delta x = \frac{3\lambda}{2} - \lambda = \frac{\lambda}{2}$$

$$\Delta t = \frac{T}{2} - 0 = \frac{T}{2}$$

$$u_p = \frac{\Delta x}{\Delta t} = \frac{\lambda}{T} \quad \text{(m/s)}$$

## To check the direction of wave propagation:

Look at $y(x,t) = A\cos\left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda} + \varphi_0\right)$ and the sign of $t$ and $x$:

If one is $\oplus$ and one $\ominus$ $\Rightarrow$ wave is traveling in $\oplus$ direction.

If both are $\oplus$ or $\ominus$ $\Rightarrow$ Wave is 〃 in $\ominus$ 〃 .

$\varphi_0$ is a constant and has no effect on diretion or speed.

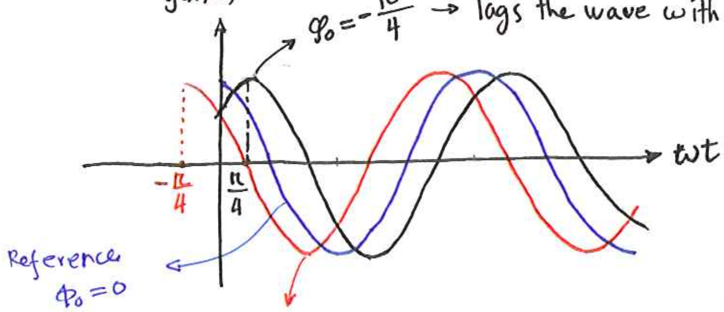
- Frequency: $\boxed{f = \frac{1}{T} \ (Hz)}$ $\rightarrow$ we can write: $u_p = \frac{\lambda}{T} = f\lambda$

- Angular velocity: $\boxed{\omega = 2\pi f \quad (rad/s)}$

- Wavenumber (or phase constant) $\boxed{\beta = \frac{2\pi}{\lambda} \quad (rad/m)}$ $\rightarrow u_p = f\lambda = \frac{f}{\frac{1}{\lambda}} = \frac{2\pi f}{\frac{2\pi}{\lambda}} = \frac{\omega}{\beta}$

So we can write $y(x,t) = A\cos\left(\underbrace{\frac{2\pi}{T}}_{\omega = 2\pi f} t - \underbrace{\frac{2\pi}{\lambda}}_{\beta} x + \varphi_0\right) = A\cos(\omega t - \beta x + \varphi_0)$

$\varphi_0 = -\frac{\pi}{4}$ $\rightarrow$ lags the wave with $\varphi_0 = 0$ by "phase lag" of $\frac{\pi}{4}$.




$\varphi_0 = \frac{\pi}{4}$ lead the wave with $\varphi_0 = 0$ by "phase lead" of $\frac{\pi}{4}$.

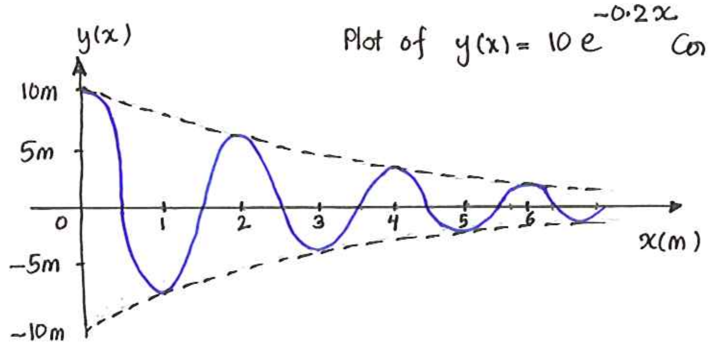
$\rightarrow \begin{cases} \Phi_0 > 0 : \text{ Phase lead in time} \\ \Phi_0 < 0 : \text{ Phase lag in time} \end{cases}$

## Sinusoidal Wave in a Lossy Medium:

$$y(x,t) = \underbrace{A e^{-\alpha x}}_{\text{Wave amplitude}} \cos(\omega t - \beta x + \Phi_0)$$

Wave amplitude attenates with $x$.

Plot of $y(x) = 10\, e^{-0.2x} \cos \pi x$




## Example

Find the expression for a sinusoidal sound wave traveling in ⊕ direction in water. frequency is 1 kHz, velocity is 1.5 km/s, the amplitude is 10 $\frac{N}{m^2}$, and the wave is max at t=0 and x = 0.25 m.

$$y(x,t) = A \cos(\omega t - \beta x + \varphi_0)$$

$$A = 10\ N/m^2$$

$$f = 1\ kHz = 1000\ Hz \rightarrow \omega = 2\pi f = 2000\pi\ rad/s$$

$$\beta = \frac{2\pi}{\lambda} \qquad \lambda = \frac{u_p}{f} = \frac{1.5\ km/s}{1\ kHz} = 1.5\ m \rightarrow \beta = \frac{2\pi}{1.5} = \frac{4\pi}{3}$$

⊕ direction => x and t must have opposite signs =>

$$y(x,t) = 10 \cos\left(2000\pi t - \frac{4\pi}{3} x + \varphi_0\right)$$

at t = 0 and x = 0.25, y(x,t) is maximum => $y(0.25, 0) = 10$

$$\Rightarrow \quad 10 \cos\left(0 - \frac{4\pi}{3} \times 0.25 + \varphi_0\right) = 10 \Rightarrow -\frac{4\pi}{3} \times 0.25 + \varphi_0 = 0 \Rightarrow \varphi_0 = \frac{\pi}{3}$$

$$\rightarrow \quad y(x,t) = 10 \cos\left(2000\pi t - \frac{4\pi}{3} x + \frac{\pi}{3}\right)$$

## Example

A laser beam of light propagating in space is given by an electric fied intensity:

$E(x,t) = 150\, e^{-0.03x} \cos(3\times10^{15} t - 10^7 x)$ (V/m). The attenuation is due to absorption in atmosphere. Determine (a) the direction of wave travel (b) wave velocity (c) wave amplitude at a distance of 200m.

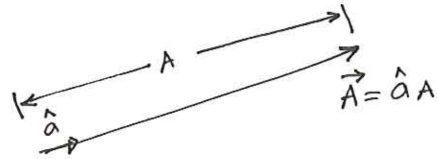
a) Direction is in ⊕ x because x & t have opposite signs.

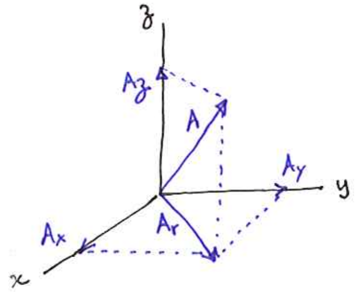
b) $u_p = \frac{\omega}{\beta} = \frac{3\times10^{15}}{10^7} = 3\times10^8$ m/s as expected (speed of light)

c) $x = 200 \rightarrow$ Amplitude $\simeq 150\, e^{-0.03\times200} = 0.37$ (V/m)

## Review of vector algebra:

vector $\vec{A} = |A|\,\hat{a}$ $\qquad \hat{a} = \dfrac{\vec{A}}{|A|}$



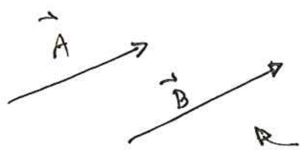

$$\vec{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$$

$$|A| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$\hat{a} = \frac{\vec{A}}{A} = \hat{x}\frac{A_x}{|A|} + \hat{y}\frac{A_y}{|A|} + \hat{z}\frac{A_z}{|A|}$$

**Equality of two vectors:** $\vec{A} = \vec{B} \Rightarrow \hat{x}A_x + \hat{y}A_y + \hat{z}A_z = \hat{x}B_x + \hat{y}B_y + \hat{z}B_z$

$$\Rightarrow \begin{cases} A_x = B_x \\ A_y = B_y \\ A_z = B_z \end{cases}$$

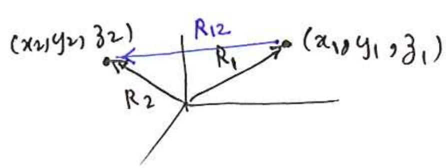



← $\vec{A}$ and $\vec{B}$ are equal but are not identical. $\vec{B}$ is displaced compared with $\vec{A}$.

**vector addition:**

$$\vec{C} = \vec{A} + \vec{B} = \vec{B} + \vec{A} = (A_x + B_x)\,\hat{a}_x + (A_y + B_y)\,\hat{a}_y + (A_z + B_z)\,\hat{a}_z$$

**Position Vector:**




$$\vec{R}_1 = \hat{x}x_1 + \hat{y}y_1 + \hat{z}z_1$$

$$\vec{R}_2 = \hat{x}x_2 + \hat{y}y_2 + \hat{z}z_2$$

**Distance vector:**

$$\vec{R}_{12} = \vec{R}_2 - \vec{R}_1 = \hat{x}(x_2 - x_1) + \hat{y}(y_2 - y_1) + \hat{z}(z_2 - z_1)$$

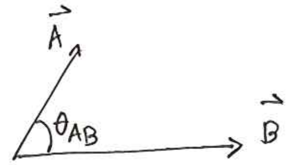
distance: $\quad d = |\vec{R}_{12}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

**vector multiplication:**

$$k\vec{A} = \hat{x}(kA_x) + \hat{y}(kA_y) + \hat{z}(kA_z)$$

## Scalar or Dot product

$$\boxed{\vec{A}\cdot\vec{B} = AB\cos\theta_{AB}}$$




$\hat{x}\cdot\hat{x} = 1$

$\hat{x}\cdot\hat{y} = 0$

$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$$

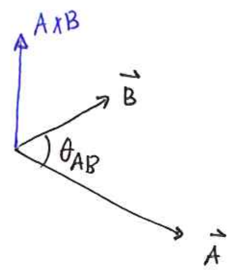
$$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$$

$$\vec{A}\cdot\vec{A} = |\vec{A}|^2 = A^2 \rightarrow |A| = \sqrt{\vec{A}\cdot\vec{A}}$$

$$\cos\theta_{AB} = \frac{\vec{A}\cdot\vec{B}}{|A||B|} \rightarrow \theta_{AB} = \cos^{-1}\left(\frac{\vec{A}\cdot\vec{B}}{\sqrt{\vec{A}\cdot\vec{A}}\,\sqrt{\vec{B}\cdot\vec{B}}}\right)$$

## Vector or scalar Product

$$\boxed{\vec{A}\times\vec{B} = \hat{n}\,AB\sin\theta_{AB}}$$




Right-hand rule

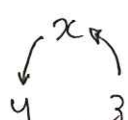
$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$$

$A\times A = 0$

$\hat{x}\times\hat{y} = \hat{z}$

$\hat{y}\times\hat{z} = \hat{x}$

$\hat{z}\times\hat{x} = \hat{y}$




$\hat{x}\times\hat{x} = 0$

$\hat{y}\times\hat{x} = -\hat{z}$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{x}\,(A_y B_z - A_z B_y) - \hat{y}\,(A_x B_z - A_z B_x) + \hat{z}\,(A_x B_y - A_y B_x)$$

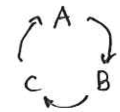
Example

Given $\vec{A} = 2\hat{x} + 3\hat{y} + 3\hat{z}$ & $\vec{B} = -\hat{x} - 5\hat{y} - \hat{z}$, what is the angle between the two vectors:

$$\theta = \cos^{-1}\frac{A\cdot B}{|A||B|} = \cos^{-1}\frac{(-2-15-3)}{\sqrt{22}\,\sqrt{27}} = 145.1^\circ$$

## Scalar Triple Product:

$$\boxed{\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})}$$

This holds if the cyclic order 

is preserved.

## Vector Triple Product:

$$\boxed{\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})}$$

"bac - cab" rule!

Note that:

$$\vec{A}(\vec{B}\cdot\vec{C}) \neq (\vec{A}\cdot\vec{B})\vec{C}$$

$$\vec{A}\times(\vec{B}\times\vec{C}) \neq (\vec{A}\times\vec{B})\times\vec{C}$$

but we have:

$$\left.\begin{aligned} \vec{A}\cdot(\vec{B}+\vec{C}) &= \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C} \\ \vec{A}\times(\vec{B}+\vec{C}) &= \vec{A}\times\vec{B} + \vec{A}\times\vec{C} \end{aligned}\right\} \text{distributive}$$

$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$ Commutative

$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$ Anticommutative

Eample:

$$\vec{A} = \hat{x} - \hat{y} + 2\hat{z} \qquad \vec{B} = \hat{y} + \hat{z} \qquad \vec{C} = -2\hat{x} + 3\hat{z}$$

find $(A\times B)\times C$ and compare with $A\times(B\times C)$.

$$A\times B = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & -1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = \hat{x}(-3) - \hat{y}(1) + \hat{z}(1) = -3\hat{x} - \hat{y} + \hat{z}$$

$$(A\times B)\times C = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -3 & -1 & 1 \\ -2 & 0 & 3 \end{vmatrix} = -3\hat{x} + 7\hat{y} - 2\hat{z}$$

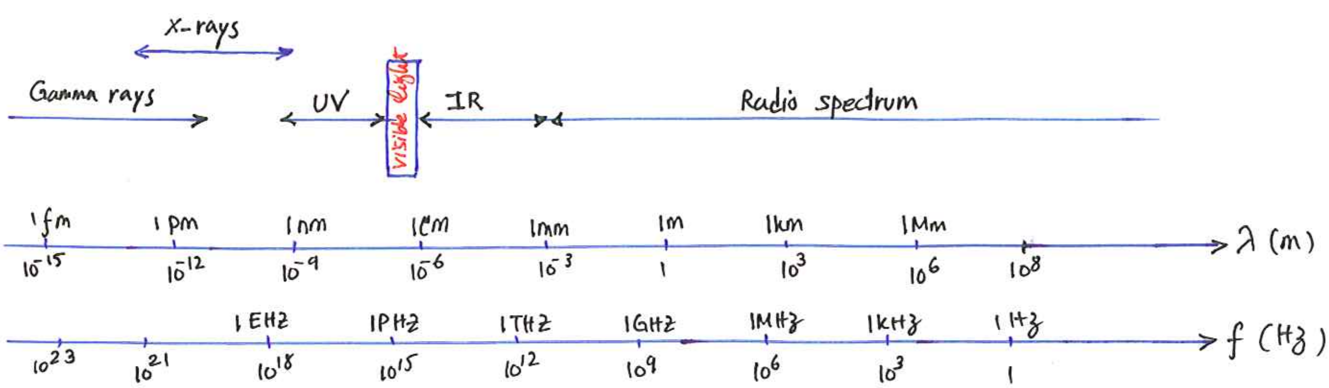
Similarly $A\times(B\times C) = 2\hat{x} + 4\hat{y} + \hat{z}$ $\rightarrow$ So they are different.

## Electromagnetic Spectrum

Visible light, X rays, gamma rays, Infr Red (IR) waves, and radio waves are all (EM) waves. They all share the following properties:

- Consist of electric and magnetic field intensities that oscillate at same frequency $f$.
- The phase velocity of EM wave in vacuum is universal constant $c = 3\times10^8$ m/s
- In vacuum, the wavelength $\lambda$ is related to $f$ by: $\lambda = \frac{c}{f}$

But they are different in frequency $f$ (or $\lambda$).

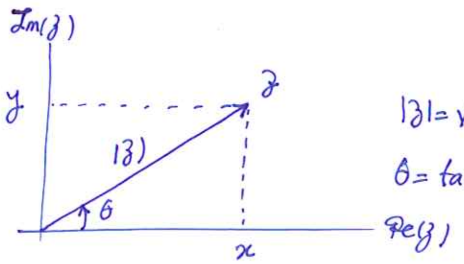



## Review of Complex Numbers:

$z = x + jy \qquad j = \sqrt{-1}$

$x = Re(z) \quad y = Im(z)$

$z = |z| e^{j\theta} = |z| \angle \theta$




$|z| = \sqrt{x^2+y^2} \qquad x = |z|\cos\theta$

$\theta = \tan^{-1}(y/x) \qquad y = |z|\sin\theta$

Euler identity: $e^{j\theta} = \cos\theta + j\sin\theta$

$z^* = (x+jy)^* = x - jy = |z|e^{-j\theta} = |z|\angle-\theta$

$|z| = \sqrt{zz^*}$

Equality of two complex numbers:

$z_1 = z_2 \implies x_1 + jy_1 = x_2 + jy_2 \implies \begin{cases} x_1 = x_2 \\ y_1 = y_2 \end{cases}$

**Addition:** $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$

**Multiplication:** $z_1 z_2 = (x_1 + jy_1)(x_2 + jy_2) = (x_1x_2 - y_1y_2) + j(x_1y_2 + x_2y_1)$

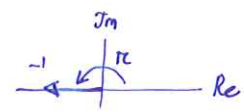
$$z_1 z_2 = |z_1| e^{j\theta_1} |z_2| e^{j\theta_2} = |z_1||z_2| e^{j(\theta_1+\theta_2)} = |z_1||z_2| \left[ \cos(\theta_1+\theta_2) + j\sin(\theta_1+\theta_2) \right]$$

**Division:** $\dfrac{z_1}{z_2} = \dfrac{x_1 + jy_1}{x_2 + jy_2} = \dfrac{(x_1+jy_1)(x_2-jy_2)}{(x_2+jy_2)(x_2-jy_2)} = \dfrac{(x_1x_2 + y_1y_2) + j(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$

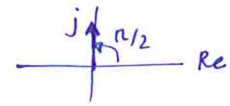
$$= \frac{|z_1| e^{j\theta_1}}{|z_2| e^{j\theta_2}} = \frac{|z_1|}{|z_2|} e^{j(\theta_1-\theta_2)} = \frac{|z_1|}{|z_2|} \left[ \cos(\theta_1-\theta_2) + j\sin(\theta_1-\theta_2) \right]$$

**Powers:** $z^n = \left(|z| e^{j\theta}\right)^n = |z|^n e^{jn\theta}$

**Useful relations:** $-1 = e^{j\pi} = e^{-j\pi} = 1\angle 180$




$j = e^{j\pi/2} = 1\angle 90$




$-j = e^{-j\pi/2} = 1\angle -90$

$$\sqrt{j} = \left(e^{j\pi/2}\right)^{1/2} = \pm e^{j\pi/4} = \pm\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right) = \pm\frac{1}{\sqrt{2}}(1+j)$$

$$\sqrt{-j} = \pm\frac{1}{\sqrt{2}}(1-j)$$

**Example**

Given $V = 3 - j4$ and $I = -(2 + j3)$

(a) Express $V$ and $I$ in polar form:

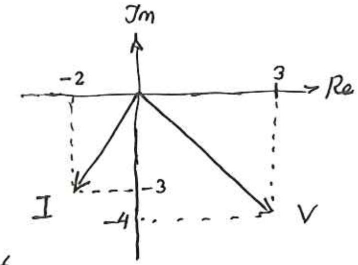
answer: $|V| = \sqrt{3^2 + 4^2} = 5$

$\theta_V = \tan^{-1}\left(\frac{-4}{3}\right) = -53.1^\circ \rightarrow V = 5e^{-j53.1}$

$|I| = \sqrt{2^2 + 3^2} = \sqrt{13} = 3.61$

$\theta_I = \tan^{-1}\frac{-3}{-2} = 180 + \tan^{-1}\frac{3}{2} = 236.3$

$I = 3.61 e^{j236.3}$




(b) $VI^* = ?$ $\quad VI = (5)(3.61)e^{-j(+53.1+236.3)}$

$= 18.05 e^{-j289.4} = 18.05 e^{j70.6}$

(C) $\sqrt{I} = ?$ $\quad \sqrt{I} = \left(3.61\, e^{j236.3}\right)^{1/2} = \pm\sqrt{3.61}\, e^{j236.3/2} = \pm 1.9\, e^{j118.15}$

## Phasors

Any cosinussoidially time-varying function $z(t)$ can be expressed as:

$$z(t) = Re\left[\tilde{Z} e^{j\omega t}\right] = \tilde{Z} \cos\omega t$$

$\tilde{Z}$ = time indepent function called phasor.

Example what is the phaser for a wave given by: $v_s(t) = V_0 \sin(\omega t + \phi_0)$

Answer: $v_s(t) = V_0 \cos\left(\frac{\pi}{2} - \omega t - \phi_0\right) = V_0 \cos\left(\omega t + \phi_0 - \frac{\pi}{2}\right)$

$$= Re\left[V_0 e^{j(\omega t + \phi_0 - \pi/2)}\right]$$

$$= Re\left[\underbrace{V_0 e^{j(\phi_0 - \pi/2)}}_{\text{Phasor}} e^{j\omega t}\right]$$

$$\tilde{V}_s = V_0 e^{j(\phi_0 - \pi/2)}$$

$i \longrightarrow \tilde{I}$

$\frac{di}{dt} \longrightarrow j\omega\tilde{I}$

$\int i\,dt \longrightarrow \frac{\tilde{I}}{j\omega}$

## Differential and Integral Equation in Phaser form:

$$\frac{di}{dt} = \frac{d}{dt}\left[Re(\tilde{I} e^{j\omega t})\right] = Re\left[\frac{d}{dt}(\tilde{I} e^{j\omega t})\right] = Re\left[j\omega\tilde{I} e^{j\omega t}\right]$$

$$\int i\,dt = \int Re\left[\tilde{I} e^{j\omega t}\right]dt = Re\left(\int \tilde{I} e^{j\omega t} dt\right) = Re\left[\frac{\tilde{I}}{j\omega} e^{j\omega t}\right]$$

So for example the phaser form of the following equation:

$$R\,i(t) + \frac{1}{C}\int i(t)\,dt = v(t) \qquad \text{(time domain)}$$

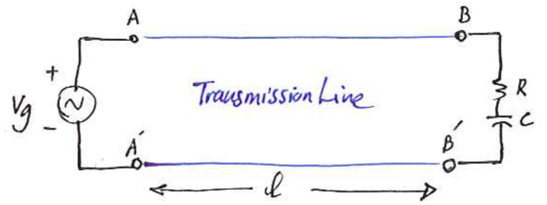
becomes:

$$R\tilde{I} + \frac{1}{j\omega C}\tilde{I} = \tilde{V} \qquad \text{(phasor domain)}$$

# Chapter 2: Transmission Lines

Generator (Source) $\longrightarrow$ [ Transmission line ] $\longrightarrow$ Load

**Example** A generator connected to an RC circuit through a transmission line of length $l$.




$V_{AA'} = V_g(t) = V_0 \cos \omega t$ what is $V_{BB'}$?

$\frac{2\pi f l}{c} = 2\pi \frac{l}{\lambda}$

Answer: $V_{BB'}(t) = V_{AA'}\left(t - \frac{l}{c}\right) = V_0 \cos\left(\omega t - \frac{\omega l}{c}\right)$

For example at $t=0$: $V_{BB'} = V_0 \cos\left(\frac{\omega l}{c}\right)$

if $f = 1\,\text{kHz}$ and $l = 5\,\text{cm}$ $\rightarrow$ $\frac{\omega l}{c} = \frac{2\pi \times 1000 \times 5 \times 10^{-2}}{3 \times 10^{8}} = 1.047 \times 10^{-6}$

$\rightarrow V_{BB'} = V_0 \cos\left(\frac{\omega l}{c}\right) = 0.\underbrace{99999999999}_{11 \text{ nines}}8\, V_0$

$\frac{\omega l}{c} = \frac{2\pi f l}{c} = 2\pi \frac{l}{\lambda}$, $\quad \frac{f}{c} = \frac{1}{\lambda}$

when $l \ll \lambda$, the transmission line effect can be ignored.

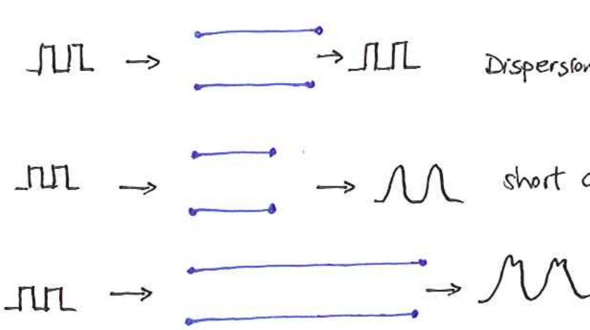
for our example $\lambda = \frac{c}{f} = \frac{3 \times 10^8}{1000} = 3 \times 10^5 \gg \underbrace{0.05}_{l}$

But when $l \gtrsim 0.01\lambda$, it may be necessary to consider two effects:

1) The phase shift due to the time delay

2) The reflected wave off from the load toward the generator.

Power loss and dispersive effects may need to be considered as well.



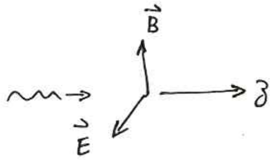
Dispersionless line

short dispersive line

Long dispersive line

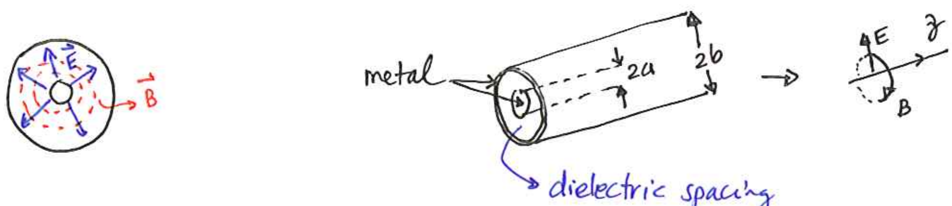
# Propagation Modes

## Transverse electromagnetic (TEM) transmission lines:




$\vec{E}$ and $\vec{B}$ are entirely transverse to the direction of propagation.

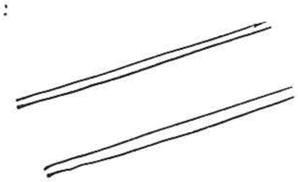
A good example is coaxial line



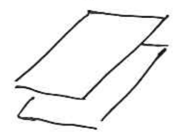

E and B are both transverse to $z$.

Other examples:

Two wire line:



Two parallel plates:

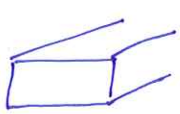


Generally TEM lines have two parallel conducting surfaces.

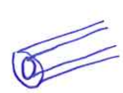
## Higher-order transmission lines:

wave propagating along these lines have at least one significant field component in the direction of propagation.

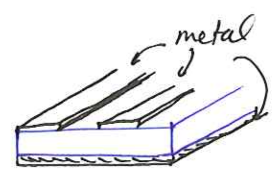
Examples:



Rectangular waveguide



optical fiber




Coplanar waveguide

# Lumped-Element Model for TEM lines

$R'$: The combined resistance of both conductors per unit length ($\Omega/m$)

$L'$: 〃 inductance 〃 ($H/m$)

$G'$: The conductance of the insulating medium per unit length ($S/m$)

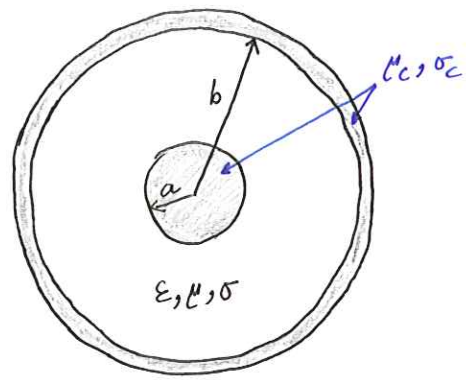
$C'$: The capacitance of the two conductors 〃 ($F/m$)

| | Coaxial | Two Wire | Parallel Plate | unit |
|---|---|---|---|---|
| $R'$ | $\frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$ | $\frac{R_s}{\pi a}$ | $\frac{2R_s}{w}$ | $(\Omega/m)$ |
| $L'$ | $\frac{\mu}{2\pi}\ln\frac{b}{a}$ | $\frac{\mu}{\pi}\ln\left[\frac{d}{2a}+\sqrt{\left(\frac{d}{2a}\right)^2-1}\right]$ | $\frac{\mu d}{w}$ | $(H/m)$ |
| $G'$ | $\frac{2\pi\sigma}{\ln(b/a)}$ | $\frac{\pi\sigma}{\ln\left[\frac{d}{2a}+\sqrt{\left(\frac{d}{2a}\right)^2-1}\right]}$ | $\frac{\sigma w}{d}$ | $(S/m)$ |
| $C'$ | $\frac{2\pi\varepsilon}{\ln(b/a)}$ | $\frac{\pi\varepsilon}{\ln\left[\frac{d}{2a}+\sqrt{\left(\frac{d}{2a}\right)^2-1}\right]}$ | $\frac{\varepsilon w}{d}$ | $(F/m)$ |

$R_s = \sqrt{\frac{\pi f \mu_c}{\sigma_c}}$ $\quad \mu_c$ and $\sigma_c$ are for the conductors.

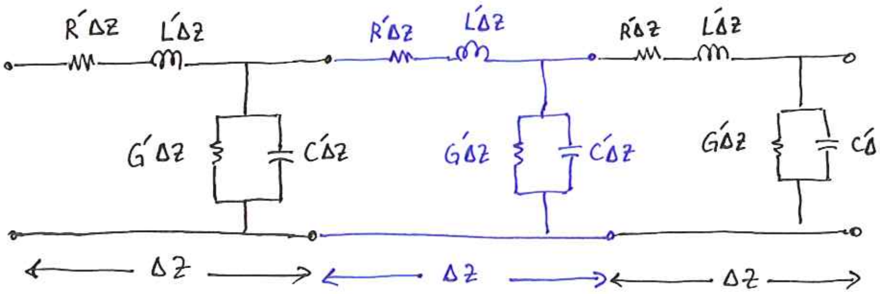
$\mu$, $\varepsilon$, and $\sigma$ for the insulating material between the conductors.

For coaxial line:

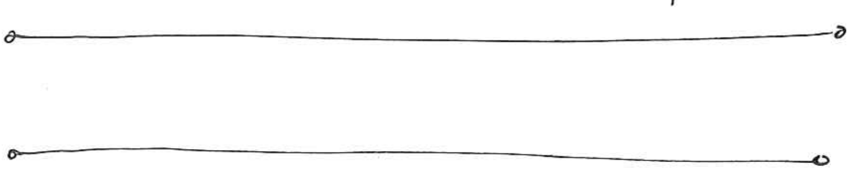



$$L'C' = \frac{\mu}{2\pi}\ln\frac{b}{a}\;\frac{2\pi\varepsilon}{\ln\frac{b}{a}} = \mu\varepsilon$$

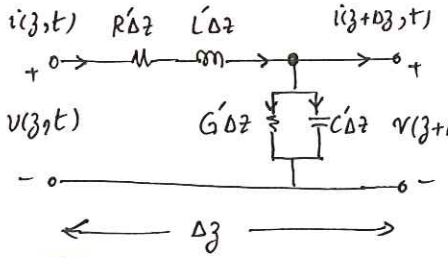
$$\frac{G'}{C'} = \frac{\sigma}{\varepsilon}$$




equivalent circuit for a transmission line



# Transmission line Equations




**Kirchhoff's voltage law:**

$$v(z,t) - R'\Delta z\, i(z,t) - L'\Delta z \frac{\partial i(z,t)}{\partial t} - v(z+\Delta z,t) = 0$$

$$\rightarrow -\frac{v(z+\Delta z,t) - v(z,t)}{\Delta z} = R' i(z,t) + L' \frac{\partial i(z,t)}{\partial t}$$

$$\Delta z \rightarrow 0 \implies \quad -\frac{\partial v(z,t)}{\partial z} = R' i(z,t) + L' \frac{\partial i(z,t)}{\partial t} \qquad (\mathrm{I})$$

**Kirchhoff's current law:**

$$i(z,t) - G'\Delta z\, v(z+\Delta z,t) - C'\Delta z \frac{\partial v(z+\Delta z)}{\partial t} - i(z+\Delta z,t) = 0$$

$$-\frac{i(z+\Delta z,t) - i(z,t)}{\Delta z} = G' v(z+\Delta z,t) + C' \frac{\partial v(z+\Delta z,t)}{\partial t}$$

$$\Delta z \rightarrow 0 \implies \quad -\frac{\partial i(z,t)}{\partial z} = G' v(z,t) + C' \frac{\partial v(z,t)}{\partial t} \qquad (\mathrm{II})$$

I and II are transmission line equations, otherwise called telegrapher's equations.

For sinusoidal steady-state conditions we may use phasor representations:

$$(\mathrm{I}) \rightarrow \boxed{\begin{aligned} -\frac{d\tilde{V}(z)}{dz} &= (R' + j\omega L')\,\tilde{I}(z) \\ -\frac{d\tilde{I}(z)}{dz} &= (G' + j\omega C')\,\tilde{V}(z) \end{aligned}}$$

Telegrapher's equation is phasor form

## Wave propagation on a Transmission Line

We can combine the two telegrapher's equations to get:

$$-\frac{d^2 V}{dz^2} = (R'+j\omega L')\frac{dI}{dz} = (R'+j\omega L')(G'+j\omega C')(-V)$$

$$\frac{d^2 V}{dz^2} - \overbrace{(R'+j\omega L')(G'+j\omega C')}^{\gamma^2} V = 0$$

$$\boxed{\frac{d^2 \tilde{V}}{dz^2} - \gamma^2 \tilde{V}(z) = 0}$$ where $\gamma^2 \equiv (R'+j\omega L')(G'+j\omega C')$

Similarly for $\tilde{I}(z)$:

$$\boxed{\frac{d^2 I}{dz^2} - \gamma^2 \tilde{I}(z) = 0}$$

wave equations

$\gamma$ is the complex propagation constant.

$\gamma$ has real and imaginary parts: $\gamma = \alpha + j\beta$

propagation constant

attenuation constant

phase constant

$$\frac{d^2 V}{dz^2} - \gamma^2 V = 0 \rightarrow V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$ → wave

$$\frac{d^2 I}{dz^2} - \gamma^2 I(z) = 0 \rightarrow I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} \quad (1)$$ → wave

Using the telegrapher's equ:

$$-\frac{dV}{dz} = (R'+j\omega L') I \rightarrow I = \frac{-1}{R'+j\omega L'}\left(-\gamma V_0^+ e^{-\gamma z} + \gamma V_0^- e^{\gamma z}\right)$$

$$\rightarrow I = \frac{\gamma}{R'+j\omega L'}\left(V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z}\right) \quad (2)$$

Comparing (1) and (2) $\Rightarrow$

$$\left.\begin{aligned} \frac{\gamma}{R'+j\omega L'} V_0^+ &= I_0^+ \\ -\frac{\gamma}{R'+j\omega L'} V_0^- &= I_0^- \end{aligned}\right\} \quad \frac{V_0^+}{I_0^+} = \frac{-V_0^-}{I_0^-} = \frac{R'+j\omega L'}{\gamma} = \sqrt{\frac{R'+j\omega L'}{G'+j\omega C'}}$$

$$Z_0 \equiv \sqrt{\frac{R'+j\omega L'}{G'+j\omega C'}} \Rightarrow \frac{V_0^+}{I_0^+} = Z_0$$

$$\frac{V_0^-}{I_0^-} = -Z_0 \rightarrow I_0^+ = \frac{V_0^+}{Z_0}, \quad I_0^- = -\frac{V_0^-}{Z_0}$$

$Z_0$ is the characteristic impedance of the line.

So we can write: $$\tilde{I}(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}$$

## Example

$L' = ?$
$C' = ?$

consider an airline where air is the dielectric between the two conductors. So $G' = 0$. Also the conductors are highly conductive so that $R' \simeq 0$. For an airline with characteristic impedance $Z_0 = 50\,\Omega$ and phase constant of 20 rad/m at 700 MHz, find the inductance per meter and the capacitance per meter of line.

- $Z_0 = 50\,\Omega$
- $\beta = 20$ rad/m at 700 MHz $\quad \beta = \mathrm{Im}(\gamma) = \mathrm{Im}\left(\sqrt{(j\omega L')(j\omega C')}\right) = \mathrm{Im}\left(j\omega\sqrt{L'C'}\right) = \omega\sqrt{L'C'}$
- $f = 700$ MHz

$$Z_0 = \sqrt{\frac{j\omega L'}{j\omega C'}} = \sqrt{\frac{L'}{C'}} \rightarrow \frac{\beta}{Z_0} = \omega C' \rightarrow C' = \frac{\beta}{\omega Z_0} = \frac{20}{2\pi \times 7 \times 10^8 \times 50}$$

$$\rightarrow C' = 9.09 \times 10^{-11}\,(\mathrm{F/m}) = 90.9\,(\mathrm{pF/m})$$

from $Z_0 = \sqrt{L'/C'} \rightarrow L' = Z_0^2 C' = (50)^2 \times 90.9 \times 10^{-12} = 2.27 \times 10^{-7}\,(\mathrm{H/m}) = 227\,(\mathrm{nH/m})$

* or use: $\beta Z_0 = \omega L' \rightarrow L' = \frac{\beta Z_0}{\omega} = \frac{(20)(50)}{2\pi \times 700 \times 10^6} = 227$ nH/m

# The Lossless Transmission Line

A transmission line has two fundamental properties: $\gamma$ and $Z_0$ ——— $Z_0, \gamma$

$\gamma$ and $Z_0$ are determined from $\omega$ (frequency), $R'$, $L'$, $G'$ and $C'$.

In many transmission lines we can choose highly conductive conductors ($R' \simeq 0$) and highly insulating dielectric ($G' \simeq 0$). If $R' \ll \omega L'$ and $G' \ll \omega C'$, we can ignore $R'$ and $G'$: $R' = G' = 0$. Therefore we can write:

$$\gamma = \alpha + j\beta = \sqrt{(\underbrace{R'}_{\text{small}} + j\omega L')(\underbrace{G'}_{\text{small}} + j\omega C')} = j\omega\sqrt{L'C'} \rightarrow \boxed{\begin{cases} \alpha = 0 & \text{lossless line} \\ \beta = \omega \underbrace{\sqrt{L'C'}}_{=\mu\varepsilon \text{ for TEM}} & \text{lossless line} \end{cases}}$$

Also for $Z_0$:

$$Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} = \sqrt{\frac{L'}{C'}} \rightarrow \boxed{Z_0 = \sqrt{\frac{L'}{C'}} \text{ lossless line}}$$

So for $\lambda$ and $u_p$ we have:

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega \underbrace{\sqrt{L'C'}}_{=\mu\varepsilon \text{ for TEM}}} \quad \text{and} \quad u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{L'C'}}$$

For a TEM line we had $L'C' = \mu\varepsilon \rightarrow \boxed{\beta = \omega\sqrt{\mu\varepsilon} \text{ (rad/m)}}$ and $\boxed{u_p = \frac{1}{\sqrt{\mu\varepsilon}} \text{ (m/s)}}$ TEM

( recall for the speed of light we had: $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ ) $\mu$ & $\varepsilon$ are for the insulator.

$\mu_0 = 4\pi \times 10^{-7}$ H/m $\quad \varepsilon_0 = 8.854 \times 10^{-12}$ F/m

For many transmission lines often $\mu = \mu_0$. But $\varepsilon = \varepsilon_r \varepsilon_0 \Rightarrow u_p = \frac{1}{\sqrt{\varepsilon_r}\sqrt{\varepsilon_0 \mu_0}} = \frac{c}{\sqrt{\varepsilon_r}}$

$$\lambda = \frac{u_p}{f} = \frac{c}{f}\frac{1}{\sqrt{\varepsilon_r}} = \frac{\lambda_0}{\sqrt{\varepsilon_r}} \rightarrow \boxed{\lambda = \frac{\lambda_0}{\sqrt{\varepsilon_r}}} \text{ when } \mu = \mu_0 \text{. TEM}$$

From $u_p = \frac{1}{\sqrt{\mu\varepsilon}}$ we see that phase velocity is independent from frequency. We call the medium nondispersive. This is true for all lossless TEM transmission lines.

For a nondispersive line all waves with different frequencies travel with same phase velocity. For example, a rectangular pulse is composed of many Fourier components with different frequencies. When a rectangular pulse enters a lossless TEM transmission line, it will arrive in the load with same shape since all of its components travel at same speed.


**Example**

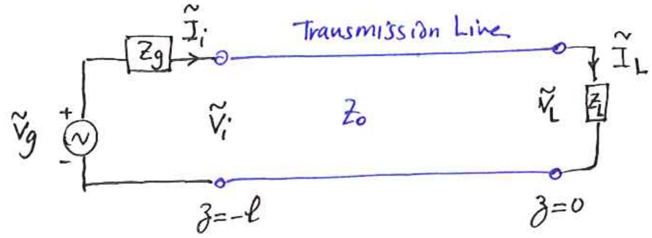
For a lossless transmission line $\lambda = 20.7$ cm at $1\,GHz$. Find $\varepsilon_r$ of the insulating material.

$$\lambda = \frac{\lambda_0}{\sqrt{\varepsilon_r}} = \frac{c}{f}\frac{1}{\sqrt{\varepsilon_r}} \rightarrow \varepsilon_r = \left(\frac{c}{f\lambda}\right)^2 = \left(\frac{3\times10^8}{10^9\times 20.7\times10^{-2}}\right)^2 = 2.1$$

## Voltage Reflection Coefficient for lossless line

We had for voltage and current:

$$\begin{cases} \tilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \\ \tilde{I}(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z} \end{cases} \xrightarrow[\gamma = j\beta]{\text{lossless line}} \begin{cases} \tilde{V}(z) = \underbrace{V_0^+ e^{-j\beta z}}_{\text{incident wave}} + \underbrace{V_0^- e^{j\beta z}}_{\text{reflected wave}} \\ \tilde{I}(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{j\beta z} \end{cases}$$




$$Z_L = \frac{\tilde{V}_L}{\tilde{I}_L} = \frac{V_0^+ + V_0^-}{\frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0}} = \left(\frac{V_0^+ + V_0^-}{V_0^+ - V_0^-}\right) Z_0 \rightarrow (V_0^+ + V_0^-) Z_0 = (V_0^+ - V_0^-) Z_L$$

$$\Rightarrow (Z_0 + Z_L) V_0^- = (Z_L - Z_0) V_0^+ \rightarrow V_0^- = \frac{Z_L - Z_0}{Z_L + Z_0} V_0^+$$

$$\boxed{\text{Reflection Coefficient} \quad \Gamma = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Z_L/Z_0 - 1}{Z_L/Z_0 + 1}}$$ (dimensionless)

$$\Gamma = \frac{V_0^-}{V_0^+} = \frac{\frac{V_0^-}{Z_0}}{\frac{V_0^+}{Z_0}} = \frac{-I_0^-}{I_0^+} \Rightarrow \boxed{\Gamma = \frac{V_0^-}{V_0^+} = \frac{-I_0^-}{I_0^+}}$$

we notice that reflection $\Gamma$ depends only on the ratio of the load impedance to $Z_0$.

Since the load impelance $Z_L$ is a complex quantity in general, $\Gamma$ may be also complex:

$$\boxed{\Gamma = \underbrace{|\Gamma|}_{\text{magnitude}}\, e^{j\overbrace{\theta_r}^{\text{Phase angle}}}}$$

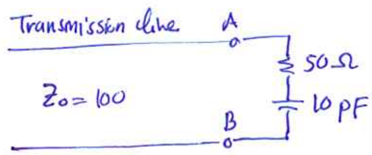
A load is matched to the line if $Z_L = Z_0 \Rightarrow \Gamma = 0 \Rightarrow$ no reflection happens.

If the load is an open circuit: $Z_L = \infty \Rightarrow \Gamma = \dfrac{Z_L/Z_0 - 1}{Z_L/Z_0 + 1} = 1 \Rightarrow V_0^- = V_0^+$ all voltage is reflected.

If the load is short circuit: $Z_L = 0 \Rightarrow \Gamma = -1 \Rightarrow V_0^- = -V_0^+ \rightarrow$ voltage is inverted and reflected.

Example

RC load:




what is the reflection coefficient at 100 MHz?

$$Z_L = R_L + \frac{1}{j\omega C_L} = 50 - \frac{j}{2\pi \times 100\times10^6 \times 10\times10^{-12}} = 50 - j\,159\ \Omega$$

$$\Gamma = \frac{Z_L/Z_0 - 1}{Z_L/Z_0 + 1} = \frac{0.5 - j1.59 - 1}{0.5 + j1.59 + 1} = \frac{-0.5 - j1.59}{1.5 + j1.59} = \frac{-1.67e^{j72.6}}{2.19e^{-j46.7}} = -0.76\,e^{j119.3^\circ}$$

we can use $-1 = e^{-j180} \Rightarrow \Gamma = 0.76\,e^{j119.3}\,e^{-j180} = 0.76\,e^{-j60.7^\circ}$

Example $|\Gamma|$ for purely reactive load.

Say $Z_L = jX_L \Rightarrow$

$$\Gamma = \frac{jX_L - Z_0}{jX_L + Z_0} = \frac{\sqrt{Z_0^2 + X_L^2}\, e^{-j\tan^{-1}\frac{X_L}{Z_0}}}{\sqrt{Z_0^2 + X_L^2}\, e^{j\tan^{-1}\frac{X_L}{Z_0}}} = e^{-j2\tan^{-1}\frac{X_L}{Z_0}}$$

$\Rightarrow |\Gamma| = 1$

# Standing waves in lossless lines

$$\left\{\begin{array}{l}\tilde{V}(z) = V_0^+ e^{-j\beta z} + \underset{\substack{\downarrow \\ \Gamma V_0^+}}{V_0^-} e^{j\beta z} = V_0^+ \left(e^{-j\beta z} + \Gamma e^{j\beta z}\right) \\ \tilde{I}(z) = \frac{V_0^+}{Z_0}\left(e^{-j\beta z} - \Gamma e^{j\beta z}\right)\end{array}\right.$$

The only unknown is $V_0^+$ is the above equations.

Let's first calculate the magnitude of $\tilde{V}(z)$:

$$|\tilde{V}(z)| = \left(\tilde{V}(z)\tilde{V}^*(z)\right)^{1/2} = \left\{ V_0^+ \left(e^{-j\beta z} + \Gamma e^{j\beta z}\right) V_0^{+*} \left(e^{j\beta z} + \Gamma^* e^{-j\beta z}\right) \right\}^{1/2}$$

$$= \left(V_0^+ V_0^{+*}\right)^{1/2} \left(1 + \Gamma\Gamma^* + \Gamma e^{j2\beta z} + \Gamma^* e^{-j2\beta z}\right)^{1/2}$$

$$= |\tilde{V}_0^+| \left(1 + |\Gamma|^2 + |\Gamma| e^{j\theta_r} e^{j2\beta z} + |\Gamma| e^{-j\theta_r} e^{-j2\beta z}\right)^{1/2}$$

$$= |\tilde{V}_0^+| \left(1 + |\Gamma|^2 + |\Gamma| \underbrace{\left(e^{j(\theta_r + 2\beta z)} + e^{-j(\theta_r + 2\beta z)}\right)}_{2\cos(\theta_r + 2\beta z)}\right)^{1/2}$$

recall: $e^x + e^{-x} = 2\cos x$

$$\boxed{|\tilde{V}(z)| = |\tilde{V}_0^+| \left[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta z + \theta_r)\right]^{1/2}}$$ voltage at each point

Similarly we can get:

$$\boxed{|\tilde{I}(z)| = \frac{|\tilde{V}_0^+|}{Z_0} \left[1 + |\Gamma|^2 - 2|\Gamma|\cos(2\beta z + \theta_r)\right]^{1/2}}$$ current at each point

Example

Plot $|\tilde{V}(z)|$ and $|\tilde{I}(z)|$ if $|\tilde{V}_0^+| = 1$, $|\Gamma| = 0.3$, $\theta_r = 30^\circ$ and $Z_0 = 50\,\Omega$

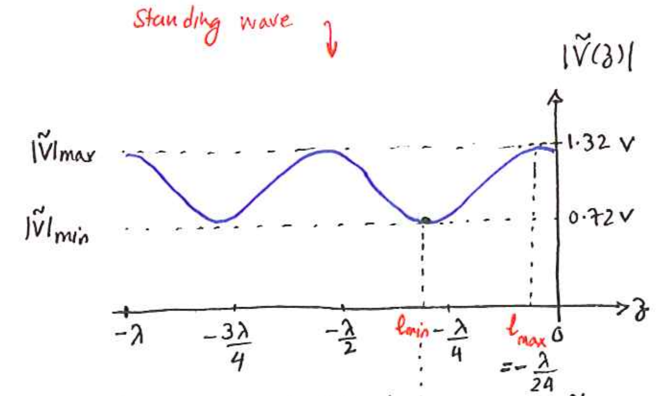
$$\Rightarrow |\tilde{V}(z)| = \left(1 + 0.09 + 0.6\cos(2\beta z + 30^\circ)\right)^{1/2}$$

$$|\tilde{I}(z)| = \frac{1}{50}\left(1 + 0.09 - 0.6\cos(2\beta z + 30^\circ)\right)^{1/2}$$

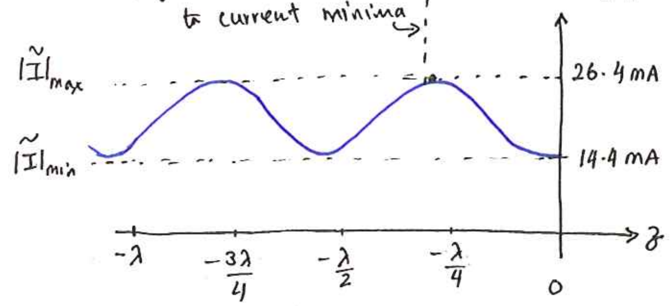
$2\beta z + 30^\circ = 0$

$z = \frac{-\pi}{6}\frac{1}{2\beta} = \frac{-\pi}{6}\frac{\lambda}{4\pi}$

$= \frac{-\lambda}{24}$




Voltage maxima correspond to current minima




The standing wave is caused by the interference of the two waves.

The maximum happens when the incident and reflected waves are in phase:

$$2\beta z + \theta_r = -2n\pi \qquad n = 0, 1, 2, \ldots$$

$$\rightarrow |\tilde{V}(z)| = |\tilde{V}_0^+| \left(1 + |\Gamma|^2 + 2|\Gamma|\right)^{1/2}$$

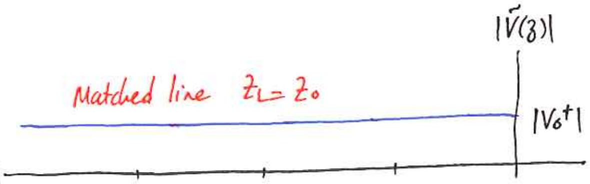
$$|\tilde{V}(z)|_{max} = |\tilde{V}_0^+|(1 + |\Gamma|) \qquad 2\beta z + \theta_r = -2n\pi$$

and $$|\tilde{V}(z)|_{min} = |\tilde{V}_0^+|(1 - |\Gamma|) \qquad 2\beta z + \theta_r = -(2n+1)\pi$$

- The repetition period for incidinent and reflected wave is $\lambda$.
- The " " " standing wave is $\frac{\lambda}{2}$.

Example 1) Matched load: $Z_L = Z_0$

$$\Rightarrow |\Gamma| = 0 \Rightarrow |\tilde{V}(z)| = |\tilde{V}_0^+|$$




2) Short circuit load: $Z_L = 0$

$$\rightarrow \Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = -1 \rightarrow \begin{cases} |V(z)|_{max} = |V_0^+|(1 + |\Gamma|) = 2|V_0^+| \\ |V(z)|_{min} = |V_0^+|(1 - |\Gamma|) = 0 \end{cases}$$

maximum happens when $2\beta z + \theta_r = -2n\pi$

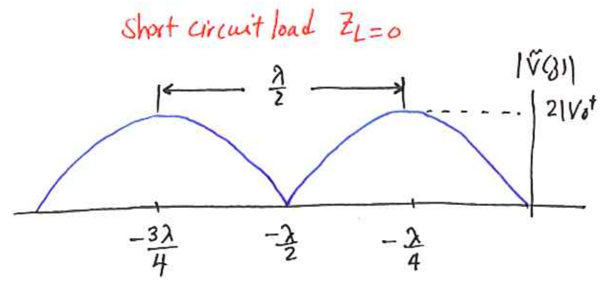
$$\rightarrow z_{max} = -\frac{\theta_r + 2n\pi}{2\beta} = -\frac{\theta_r + 2n\pi}{\frac{4\pi}{\lambda}} = -\frac{\theta_r \lambda}{4\pi} - \frac{n\lambda}{2}$$

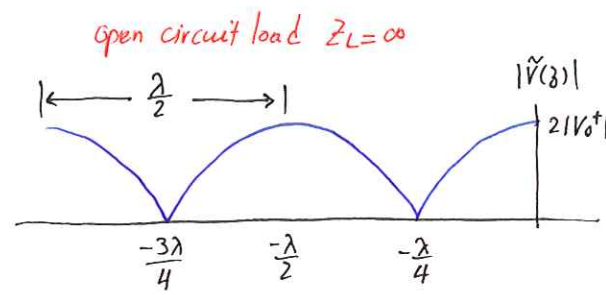
$$\Gamma = -1 \Rightarrow \theta_r = \pi \Rightarrow z_{max} = -\frac{\lambda}{4} - \frac{n\lambda}{2}$$

$z_{min}$ happens when $2\beta z + \theta_r = -(2n+1)\pi$

$$\rightarrow z_{min} = -\frac{\theta_r + 2n\pi + \pi}{2\beta} = -\frac{\theta_r \lambda}{4\pi} - \frac{n\lambda}{2} - \frac{\lambda}{4}$$

$$\rightarrow \boxed{z_{min} = z_{max} - \frac{\lambda}{4}} \quad \text{or} \quad \boxed{z_{min} = z_{max} + \frac{\lambda}{4}}$$ check the line




3) Open circuit load: $Z_L = \infty \rightarrow \Gamma = 1$, $\theta_r = 0$ $\rightarrow \begin{cases} z_{max} = -\frac{n\lambda}{2} \\ z_{min} = -\frac{n\lambda}{2} - \frac{\lambda}{4} \end{cases}$

voltage standing-wave ratio: $S$ (VSWR) or (SWR)

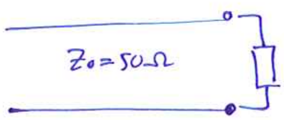
$$\boxed{S \equiv \frac{|\tilde{V}|_{max}}{|\tilde{V}|_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}} \quad \text{(dimensionless)}$$

SWR provides a measure of the mismatch between the load and the line.

For a matched load $|\Gamma|=0$ and we get $S=1$.

for a short circuit or open circuit that has the maximum mismatch we have $|\Gamma|=1$ so $S=\infty$.

Example

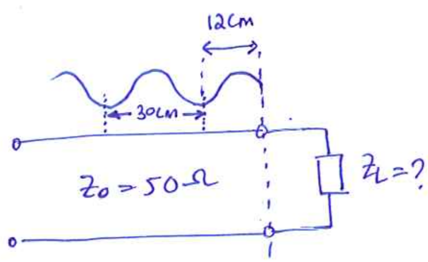



what is SWR? must find $\Gamma$ first)

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(100+j50)-50}{(100+j50)+50} = \frac{50+j50}{150+j50} = \frac{70.7e^{j45}}{158.1e^{j18.4}} = 0.45e^{j26.6}$$

$$S = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.45}{1-0.45} = 2.6$$

Example




$S=3$

$l_{min}=12$ cm

From $S \longrightarrow |\Gamma|$

from $l_{min}$ & $\beta \longrightarrow \theta_r$

$\Big\} \rightarrow \Gamma$, $Z_0$ $\Big\} \rightarrow Z_L$

$$\frac{\lambda}{2} = 30 \rightarrow \lambda = 60 \text{ cm}$$

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi}{0.6\text{ m}} = \frac{10\pi}{3} \quad (\text{rad/m})$$

$$S = \frac{1+|\Gamma|}{1-|\Gamma|} \rightarrow |\Gamma| = \frac{S-1}{S+1} = \frac{3-1}{3+1} = 0.5$$

for minimum: $2\beta z_{min} + \theta_r = -(2n+1)\pi$

$$(2)\left(\frac{10\pi}{3}\right)(-0.12) + \theta_r = -\pi \rightarrow \theta_r = -\pi + \left(\frac{20\pi}{3}\right)(0.12) = -0.2\pi = -36^\circ$$

$$\rightarrow \Gamma = |\Gamma|e^{j\theta_r} = 0.5e^{-j36} = 0.405 - j0.294 \qquad \Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} \Rightarrow Z_L = Z_0\frac{1+\Gamma}{1-\Gamma} \Rightarrow$$

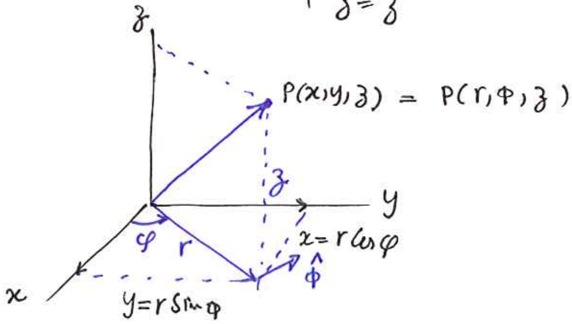
$$Z_L = (50)\left(\frac{1.405 - j0.294}{1-0.405+j0.294}\right) = 85 - j67 \ \Omega$$

## 3-3 Transformation between Coordinate Systems

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{cases}$$

### Cartesian to Cylindrical:

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \varphi = \tan^{-1}\dfrac{y}{x} \\ z = z \end{cases}$$




$\hat{r}\cdot\hat{x} = \cos\varphi$

$\hat{r}\cdot\hat{y} = \cos(\frac{\pi}{2}-\varphi) = \sin\varphi$

$\hat{\phi}\cdot\hat{x} = -\sin\varphi$

$\hat{\phi}\cdot\hat{y} = \cos\varphi$

$$\hat{r} = \hat{x}a + \hat{y}b$$

$$\hat{r}\cdot\hat{x} = a + \underbrace{\cancel{\hat{y}\cdot\hat{x}}}_{0}\,b \rightarrow a = \cos\varphi$$

$$\hat{r}\cdot\hat{y} = \underbrace{\cancel{\hat{x}\cdot\hat{y}}}_{0}\,a + b \rightarrow b = \sin\varphi$$

$$\rightarrow \boxed{\hat{r} = \hat{x}\cos\varphi + \hat{y}\sin\varphi} \quad (\mathrm{I})$$

$$\hat{\varphi} = \hat{x}a + \hat{y}b$$

$$\hat{\varphi}\cdot\hat{x} = a \rightarrow a = -\sin\varphi$$

$$\hat{\phi}\cdot\hat{y} = b \rightarrow b = \cos\varphi$$

$$\rightarrow \boxed{\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi} \quad (\mathrm{II}) \qquad \boxed{\hat{z} = \hat{z}}$$

$$(\mathrm{I})\ \&\ (\mathrm{II}) \rightarrow \boxed{\begin{aligned} \hat{x} &= \hat{r}\cos\varphi - \hat{\phi}\sin\varphi \\ \hat{y} &= \hat{r}\sin\varphi + \hat{\phi}\cos\varphi \end{aligned}}$$

To convert a vector $\vec{A} = \hat{r}A_r + \hat{\phi}A_\phi + \hat{z}A_z$ to cartesian coordinate:

$$\vec{A} = (\hat{x}\cos\varphi + \hat{y}\sin\varphi)A_r + (-\hat{x}\sin\varphi + \hat{y}\cos\varphi)A_\varphi + \hat{z}A_z$$

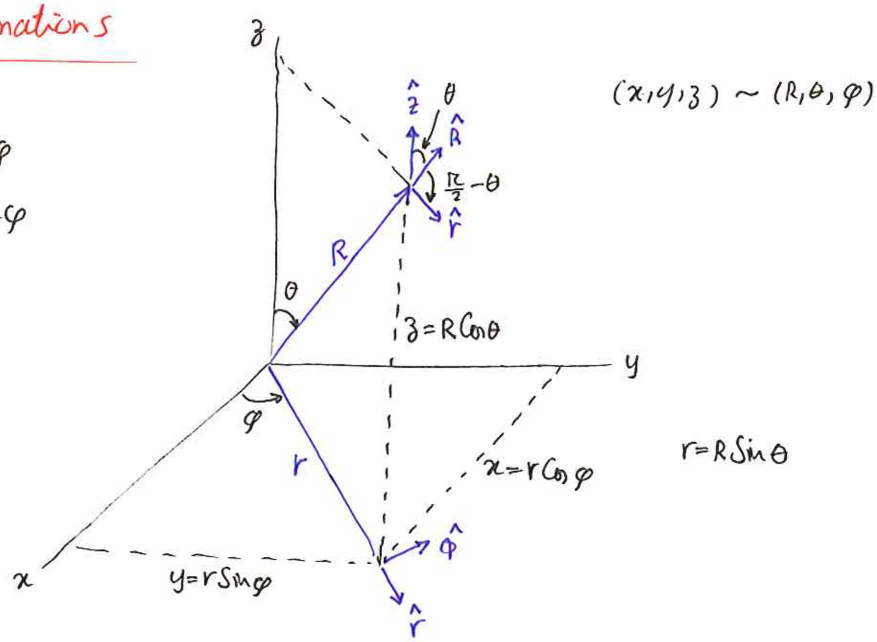
$$= (A_r\cos\varphi - A_\varphi\sin\varphi)\hat{x} + (A_r\sin\varphi + A_r\cos\varphi)\hat{y} + A_z\hat{z}$$

$$\rightarrow \boxed{\begin{aligned} A_x &= A_r\cos\varphi - A_\varphi\sin\varphi \\ A_y &= A_r\sin\varphi + A_r\cos\varphi \end{aligned}}$$

and conversly $\boxed{\begin{aligned} A_r &= A_x\cos\phi + A_y\sin\phi \\ A_\phi &= -A_x\sin\phi + A_y\cos\phi \end{aligned}}$

# Cartesian To Spherical Transformations

$$\begin{cases} R=\sqrt{x^2+y^2+z^2} \\ \theta=\tan^{-1}\dfrac{\sqrt{x^2+y^2}}{z} \\ \varphi=\tan^{-1}\dfrac{y}{x} \end{cases} \qquad \begin{cases} x=R\sin\theta\cos\varphi \\ y=R\sin\theta\sin\varphi \\ z=R\cos\theta \end{cases}$$




$\hat{R}$ is in $\hat{r}$-$\hat{z}$ plane. So:

$$\hat{R}=\hat{r}a+\hat{z}b$$

$$\hat{R}\cdot\hat{r}=a+\cancelto{0}{\hat{z}\cdot\hat{r}}\,b \rightarrow a=\hat{R}\cdot\hat{r}=\sin\theta$$

$$\hat{R}\cdot\hat{z}=\cancelto{0}{\hat{r}\cdot\hat{z}}\,a+b \rightarrow b=\hat{R}\cdot\hat{z}=\cos\theta$$

$$\rightarrow \hat{R}=\hat{r}\sin\theta+\hat{z}\cos\theta=\hat{x}\cos\varphi\sin\theta+\hat{y}\sin\phi\sin\theta+\hat{z}\cos\theta$$

$$\boxed{\hat{R}=\hat{x}\cos\varphi\sin\theta+\hat{y}\sin\phi\sin\theta+\hat{z}\cos\theta}$$

Similarly for $\hat{\theta}$: $\boxed{\hat{\theta}=\hat{x}\cos\theta\cos\phi+\hat{y}\cos\theta\sin\phi-\hat{z}\sin\theta}$

and for $\hat{\phi}$: $\boxed{\hat{\phi}=-\hat{x}\sin\phi+\hat{y}\cos\phi}$

from these three we can extract $\hat{x},\hat{y},\hat{z}$:

$$\boxed{\begin{aligned} \hat{x}&=\hat{R}\sin\theta\cos\phi+\hat{\theta}\cos\theta\cos\phi-\hat{\phi}\sin\phi \\ \hat{y}&=\hat{R}\sin\theta\sin\phi+\hat{\theta}\cos\theta\sin\phi+\hat{\phi}\cos\phi \\ \hat{z}&=\hat{R}\cos\theta-\hat{\theta}\sin\theta \end{aligned}}$$

## Cylinrical to Spherical Transformation

It can be realized by combining the transformation between Cartesian & cylindrical and Cartesian & Spherical. See table 3-2 in the book.

## Distance between two points

$P_1(x_1, y_1, z_1) \qquad P_2(x_2, y_2, z_2)$

In Cartesian: $\quad d$

$$d = |R_{12}| = \left[ (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 \right]^{1/2}$$

In cylindrical:

$$d = \left[ (r_1 \cos\varphi_1 - r_2 \cos\varphi_2)^2 + (r_1 \sin\varphi_1 - r_2 \sin\varphi_2)^2 + (z_1 - z_2)^2 \right]^{1/2}$$

In spherical:

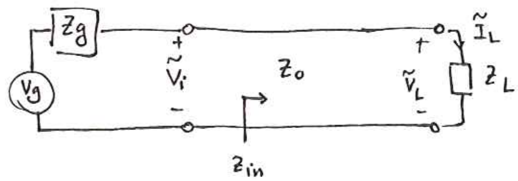
$$d = \left[ (R_1 \sin\theta_1 \cos\varphi_1 - R_2 \sin\theta_2 \cos\varphi_2)^2 + (R_1 \sin\theta_1 \sin\varphi_1 - R_2 \sin\theta_2 \sin\varphi_2)^2 + (R_1 \cos\theta_1 - R_2 \cos\theta_2)^2 \right]^{1/2}$$

$$d = \left\{ R_1^2 + R_2^2 - 2R_1R_2 \left[ \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\varphi_2 - \varphi_1) \right] \right\}^{1/2}$$

## Input Impedance of the Lossless Line

$$Z_{in}(z) = \frac{\tilde{V}(z)}{\tilde{I}(z)} = \frac{V_0^+ \left(e^{-j\beta z} + \Gamma e^{j\beta z}\right)}{\frac{V_0^+}{Z_0}\left(e^{-j\beta z} - \Gamma e^{j\beta z}\right)}$$

$$= Z_0 \frac{1+\Gamma e^{j2\beta z}}{1-\Gamma e^{j2\beta z}}$$




Note that $Z_{in}$ is the ratio of the total voltage to tatal current. Otherwise the ratio of say incident voltage to the incident current is $Z_0$: $Z_0 = \frac{V_0^+}{I_0^+} = \frac{-V_0^-}{I_0^-}$

At in input of the line $z = -\ell \Rightarrow$

$$Z_{in}(-\ell) = Z_0 \frac{e^{j\beta\ell} + \Gamma e^{-j\beta\ell}}{e^{j\beta\ell} - \Gamma e^{-j\beta\ell}}$$

$$= Z_0 \frac{e^{j\beta\ell} + \Gamma e^{-j\beta\ell}}{e^{j\beta\ell} - \Gamma e^{-j\beta\ell}} = Z_0 \frac{e^{j\beta\ell} + \frac{Z_L - Z_0}{Z_L + Z_0} e^{-j\beta\ell}}{e^{j\beta\ell} - \frac{Z_L - Z_0}{Z_L + Z_0} e^{-j\beta\ell}}$$

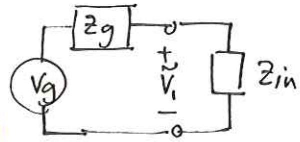
$$= Z_0 \frac{(Z_L+Z_0)e^{j\beta\ell} + (Z_L - Z_0)e^{-j\beta\ell}}{(Z_L+Z_0)e^{j\beta\ell} - (Z_L - Z_0)e^{-j\beta\ell}}$$

$$e^{jx} = \cos x + j\sin x$$

$$= Z_0 \frac{(Z_L+Z_0)(\cos\beta\ell + j\sin\beta\ell) + (Z_L - Z_0)(\cos\beta\ell - j\sin\beta\ell)}{(Z_L+Z_0)(\cos\beta\ell + j\sin\beta\ell) - (Z_L - Z_0)(\cos\beta\ell - j\sin\beta\ell)}$$

$$= Z_0 \frac{Z_L\cos\beta\ell + jZ_0\sin\beta\ell}{Z_0\cos\beta\ell + jZ_L\sin\beta\ell}$$

$$\boxed{Z_{in}(\ell) = \frac{Z_L + jZ_0\tan\beta\ell}{Z_0 + jZ_L\tan\beta\ell}}$$



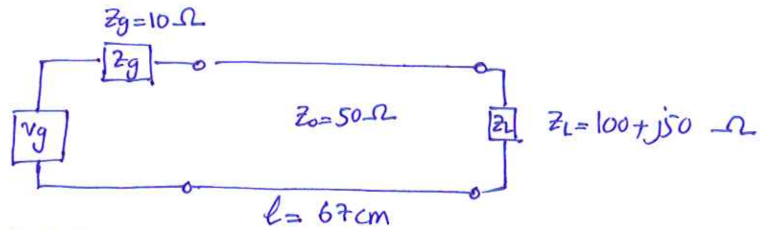

So we can write $\tilde{V}_i = \tilde{I}_i Z_{in} = \tilde{V}_g \frac{Z_{in}}{Z_{in}+Z_g}$

we also know that $\tilde{V}_i = \tilde{V}(-\ell) = V_0^+ e^{j\beta\ell} + V_0^- e^{-j\beta\ell}$

$$= V_0^+\left(e^{j\beta\ell} + \Gamma e^{-j\beta\ell}\right)$$

$$\boxed{V_0^+ = \frac{\tilde{V}_g Z_{in}}{Z_{in}+Z_g} \frac{1}{e^{j\beta\ell} + \Gamma e^{-j\beta\ell}}}$$

Example




$v_g(t) = 10\sin(\omega t + 30)$

$f = 1.05$ GHz

$u_p = 0.7c$ phase velocity

Find $v(z,t)$ and $i(z,t)$ on the line?

Solution: $\tilde{V}(z) = \underset{?}{\underset{\downarrow}{V_0^+}}\left(\underset{?}{\underset{\downarrow}{e^{-j\beta z}}} + \underset{?}{\underset{\downarrow}{\Gamma}}\, e^{j\beta z}\right)$

$\beta = \frac{2\pi}{\lambda} \rightarrow ?$

Solution: $\lambda = \frac{u_p}{f} = \frac{0.7 \times 3 \times 10^8}{1.05 \times 10^9} = 0.2\,\text{m} \rightarrow \beta = \frac{2\pi}{\lambda} = \frac{2\pi}{0.2} = 10\pi$

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan\beta\ell}{Z_0 + jZ_L \tan\beta\ell}$$

$$= Z_0 \frac{Z_L/Z_0 + j\tan\beta\ell}{1 + j(Z_L/Z_0)\tan\beta\ell} = 50\, \frac{2 + j + j\overbrace{\tan(6.7\pi)}^{=\tan(0.7\pi) = \tan 126^\circ}}{1 + j(2+j)\tan(6.7\pi)} = 21.9 + j17.4\ \Omega$$

Need phasors for $v_g(t)$:

$v_g(t) = 10\sin(\omega t + 30) = 10\cos(\frac{\pi}{2} - \omega t - 30) = 10\cos(\omega t - 60^\circ) \rightarrow \tilde{V}_g = 10e^{-j60^\circ} = 10\angle{-60^\circ}$

$$V_0^+ = \frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}} \frac{1}{e^{j\beta\ell} + \Gamma e^{-j\beta\ell}}$$

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{100 + j50 - 50}{100 + j50 + 50} = 0.45e^{j26.6^\circ}$$

$$\rightarrow V_0^+ = \frac{10e^{-j60^\circ}(21.9 + j17.4)}{10 + 21.9 + j17.4} \times \frac{1}{e^{j126} + \underbrace{0.45e^{j26.6}}_{\Gamma}\, e^{-j126}} = 10.2e^{j159}\ (\text{V})$$

$$\tilde{V}(z) = V_0^+(e^{-j\beta z} + \Gamma e^{j\beta z}) = 10.2e^{j159}(e^{-j\beta z} + 0.45e^{j26.6} e^{j\beta z})$$

$$= 10.2\, e^{-j\beta z} e^{j159} + 4.55 e^{j\beta z} e^{185.6}$$

$$\rightarrow v(z,t) = 10.2\cos(\omega t - \overset{10\pi}{\overset{\uparrow}{\beta}} z + 159^\circ) + 4.55\cos(\omega t + \overset{10\pi}{\overset{\uparrow}{\beta}} z + 185.6) \quad \text{where } \beta = 10\pi$$

$$\tilde{I}(z) = \frac{V_0^+}{Z_0}(e^{-j\beta z} - \Gamma e^{j\beta z}) \longrightarrow i(z,t) = 0.2\cos(\omega t - \beta z + 159) + 0.091\cos(\omega t + \beta z + 5.6)$$